Link-Disjoint Regular Spanning Subnetworks of Hypercubes

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Abstract:

When Q_n, a hypercube of dimension n, is used as the basis of a computing environment with multiple users, users traditionally request a complete hypercube of smaller dimension. Using this scheme, only one user could have access to a hypercube of dimension n at any given time.

We propose that the nodes of Q_n be divided into k virtual nodes, where $2 \leq k \leq n / 2$, and the links of Q_n be divided into k disjoint sets $S_1, S_2, ..., S_k$ of $n2^{n-1} / k$ links such that the nodes of Q_n and the links of any S_i form a regular spanning subnetwork of Q_n. This way, k different users could have access to a virtual copy of Q_n at any given time. The sets $S_1, S_2, ..., S_k$ are constructed initially by choosing link-disjoint Hamilton cycles (LDHC's) of Q_n, then combining portions of LDHC's to create networks of small diameter. It is known that Q_n has $\lfloor n / 2 \rfloor$ LDHC's for all n [10]. In this paper we demonstrate the following:

- a specific method for generating n / 2 LDHC's in Q_n, where $n = 2^k$ or $n = 3 * 2^k, k > 1$
- a method for constructing 2 link-disjoint spanning subnetworks (LDSS's) of Q_n of degree 3, from any 3 LDHC's of Q_n, with experimental results for Q_6.
- a method for constructing 2 LDSS's of Q_n, when n is a power of 2, where each LDSS has degree n / 2 and diameter 2n
- a method for constructing n / 4 LDSS's of Q_n, when n is a power of 2, where each LDSS has degree 4 and diameter $2^{n/2}$.

Keywords: Hypercubes, link deletion, link-disjoint Hamilton cycles, spanning subnetworks, diameter, routing.

I. Introduction

In traditional computing environments with a single processor, a large number of memory locations and multiple users, memory is considered to be a partitionable resource. Processes make requests for certain amounts of contiguous memory locations, and various allocation and collection strategies have been adopted in an attempt to minimize memory fragmentation. In an environment with 2^n processors, connected in the form of the hypercube Q_n, and multiple users, the body of processors is also considered to a partitionable resource. Q_n is known to have a recursive substructure, that is, Q_n is comprised of 2 copies of Q_{n-1}, with links connecting corresponding nodes in the two copies. Therefore, when processes make requests for some of the processors, they usually request a complete hypercube of dimension smaller than n, commonly called a subcube. Significant research has been conducted into identifying, allocating, and recollecting subcubes [8], [11], [4]. In this environment, if two or more processes require a hypercube of dimension n, only one of these processes will be able to run at any given time.

We propose to expand the effective capacity of Q_n as a partitionable resource by changing the way the nodes are partitioned, and partitioning the *links* of Q_n as well. Currently, when a process asks to use some of the processors of Q_n, it gets full control of all the processors assigned to it. We propose that each node of Q_n be divided into k *virtual nodes*, where $2 \le k \le n / 2$. We propose that the links of Q_n be partitioned into k disjoint sets $S_1, S_2, ..., S_k$, of $n2^{n-1} / k$ links each where the members of any S_i and the nodes of Q_n form a regular spanning subnetwork of Q_n. The j^{th} virtual node of any node in Q_n will be connected to the j^{th} virtual node of other nodes in Q_n by the members of S_j. Under this arrangement, up to k different processes can have access to a virtual copy of Q_n at the same time with no interference between computations, since communication between virtual nodes is taking place over disjoint link sets. Figure 1 shows an instance of this arrangement, where n = 4 and k = 2. S_1 is the set of gray links, connecting the gray virtual nodes, and S_2 is the set of black links, connecting the black virtual nodes.

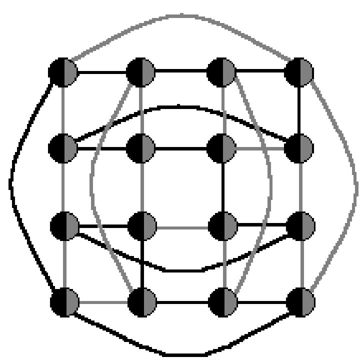

Figure 1. Two link-disjoint regular spanning subnetworks of Q_4

We believe that the sets $S_1, S_2, ..., S_k$ can be selected so that the resulting spanning subnetworks of Q_n have small diameter. Q_n has a node-connectivity of n, which means any two vertices in Q_n can be connected by n node-disjoint paths [6]. Therefore, perhaps some, or even most of the links in Q_n can be removed without increasing its diameter. This has been a subject of recent research [5],

[3], [7]. It has been shown that $(n-2)2^{n-1}+1-\left\lceil \frac{2^n-1}{2n-1} \right\rceil$ links can be removed from Q_n without increasing its diameter [3]. Table 1 shows the number of links that can be removed from Q_n for small values of n.

n	2	3	4	5	6	7	8
Links in Q_n	4	12	32	80	192	448	1024
Removable Links	0	3	14	45	123	311	752

Table 1. The number of links that can be removed from Qn without increasing the diameter.

Unfortunately, the resulting spanning subnetwork of Q_n is not regular. Furthermore, the deleted links do not form a spanning subnetwork of Q_n. Regular spanning subnetworks exist for $Q_{n + \lg n}$ and $Q_{n - 1}$, where $n = 2^k$, $k > 1$, [2], [1]. But again, the links of the respective hypercubes which are not part of the above subnetworks do not form spanning subnetworks. In the following section, we will identify n / 2 link-disjoint regular spanning subnetworks in Q_n, for selected values of n.

II. Generating n / 2 Link-Disjoint Hamilton Cycles (LDHC's) in Q_n

Q_n is known to have a very large number of Hamilton cycles [9], [10]. Let C be a Hamilton cycle in Q_n. Let *C(k)* be the k^{th} node in C ($0 \le k \le 2^n - 1$). Let $Q_{2n}(n, k)$ be the k^{th} copy of Q_n inside Q_{2n} ($0 \le k \le 2^n - 1$). Let $Q_{2n}(n, k, m)$ be the m^{th} node in $Q_{2n}(n, k)$ ($0 \le m \le 2^n - 1$). Let the first n bits of $Q_{2n}(n, k, m)$ be C(k). Let the last n bits of $Q_{2n}(n, k, m)$ be C(m). We observe that by the definition of $Q_{2n}(n, k, m)$, there is a link between $Q_{2n}(n, k, m)$ and $Q_{2n}(n, (k \pm 1) \bmod 2^n, m)$, for all k and m. There is also a link between $Q_{2n}(n, k, m)$ and $Q_{2n}(2, k, (m \pm 1) \bmod 2^n)$, for all k and m. Let the *left child of C* or *LC(C)* be a sequence of nodes in Q_{2n}. Let *LC(C)(k)* be the k^{th} node in LC(C) ($0 \le k \le 2^{2n} - 1$). Let $LC(C)(k) = Q_{2n}(n, k \text{ div } 2^n, ((k \bmod 2^n) - (k \text{ div } 2^n)) \bmod 2^n)$, where div is integer division with no remainder.

Lemma 1: If C is a Hamilton cycle in Q_n, then LC(C) is a Hamilton cycle in Q_{2n}.

Proof: If $k \bmod 2^n = 0$, then LC(C)(k), LC(C)(k + 1), ..., LC(C)($k + 2^n - 1$) are the 2^n nodes of $Q_{2n}(n, k \text{ div } 2^n)$. LC(C) traverses all of $Q_{2n}(n, 0)$, then all of $Q_{2n}(n, 1)$, ..., then all of $Q_{2n}(n, 2^n - 1)$. LC(C)(0) is the node C(0) | C(0) (C(0) concatenated with C(0)). LC(C)($2^{2n} - 1$) is the node C($2^n - 1$) | C(0). There is a link in Q_{2n} between LC(C)(0) and LC(C)($2^{2n} - 1$). Therefore LC(C) is a Hamilton cycle in Q_{2n}. ν

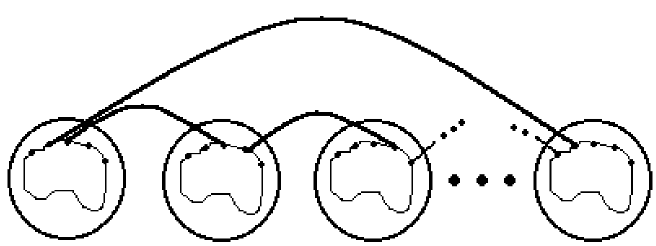

Figure 2. The construction of LC(C), a Hamilton cycle in Q_{2n}.

Let the *right child of C* or *RC(C)* be a sequence of nodes in Q_{2n}. Let *RC(C)(k)* be the k^{th} node in RC(C) ($0 \le k \le 2^{2n} - 1$). Let $RC(C)(k) = Q_{2n}(n, ((k \bmod 2^n) - (k \text{ div } 2^n)) \bmod 2^n, k \text{ div } 2^n)$.

Lemma 2: If C is a Hamilton cycle in Q_n, then RC(C) is a Hamilton cycle in Q_{2n}.

Proof: If k mod $2^n = 0$, then the sequence RC(C)(k), RC(C)(k + 1), ..., RC(C)(k + 2^n - 1) traverses the (k div 2^n)th node of each Q_{2n}(n, j), ($0 \leq j \leq 2^n - 1$). RC(C)(0) is the node C(0) | C(0). RC(C)(2^{2n} - 1) is the node C(0) | C(2^n - 1). Therefore RC(C) is a Hamilton cycle of Q_{2n}. ν

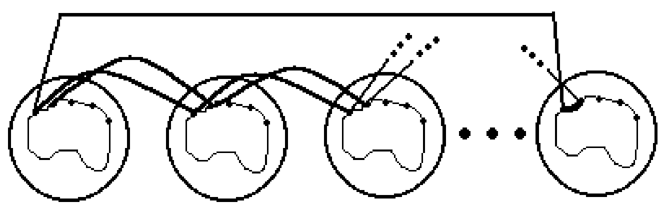

Figure 3. The construction of RC(C), another Hamilton cycle in Q_{2n}.

Lemma 3: If C is a Hamilton cycle in Q_n, then LC(C) and RC(C) are LDHC's in Q_{2n}.

Proof: Assume LC(C) and RC(C) are not link-disjoint. Then there exist k and m where LC(C)(k) = RC(C)(m) and LC(C)((k + 1) mod 2^{2n}) = RC(C)((m + 1) mod 2^{2n}). If this is true, then the following three equations are true by the definitions of LC(C) and RC(C):

(1) ((k mod 2^n) - (k div 2^n)) mod 2^n = m div 2^n

(2) k div 2^n = ((m mod 2^n) - (m div 2^n)) mod 2^n

(3) ((((k + 1) mod 2^{2n}) mod 2^n) - (((k + 1) mod 2^{2n}) div 2^n)) mod 2^n = ((m + 1) mod 2^{2n}) div 2^n

Equations (1) and (2) can be solved for m in terms of k.

m = (((k mod 2^n) - (k div 2^n)) mod 2^n)2^n + ((((k mod 2^n) - (k div 2^n)) mod 2^n) + (k div 2^n)) mod 2^n

Since the first term in this equation is a multiple of 2^n,

$$m \bmod 2^n = ((((k \bmod 2^n) - (k \operatorname{div} 2^n)) \bmod 2^n) + (k \operatorname{div} 2^n)) \bmod 2^n$$

Which simplifies to...

$$m \bmod 2^n = k \bmod 2^n$$

Let *a* represent the expression obtained by subtracting the left side of equation (1) from the left side of equation (3):

$$(((((k + 1) \bmod 2^{2n}) \bmod 2^n) - (((k + 1) \bmod 2^{2n}) \operatorname{div} 2^n)) \bmod 2^n) -$$
$$(((k \bmod 2^n) - (k \operatorname{div} 2^n)) \bmod 2^n)$$

If $k \bmod 2^n = 2^n - 1$, then $a = 0$. If $k \bmod 2^n < 2^n - 1$ and $k \bmod 2^n \neq (k \operatorname{div} 2^n) - 1$, then $a = 1$. If $k \bmod 2^n < 2^n - 1$ and $k \bmod 2^n = (k \operatorname{div} 2^n) - 1$, then $a = -(2^n - 1)$. Let *b* represent the expression obtained from subtracting the right side of equation (1) from the right side of equation (3)

$$(((m + 1) \bmod 2^{2n}) \operatorname{div} 2^n) - (m \operatorname{div} 2^n)$$

If $m \bmod 2^n < 2^n - 1$, then $b = 0$. If $m \bmod 2^n = 2^n - 1$ and $m \operatorname{div} 2^n < 2^n - 1$, then $b = 1$. If $m \bmod 2^n = 2^n - 1$ and $m \operatorname{div} 2^n = 2^n - 1$, then $b = -(2^n - 1)$. If LC(C) and RC(C) are not link-disjoint, then there exists k which a = b. If $k \bmod 2^n = 2^n - 1$, then $m \bmod 2^n = 2^n - 1$. Then a = 0 and b is either 1 or $-(2^n - 1)$. If $k \bmod 2^n < 2^n - 1$, then $m \bmod 2^n < 2^n - 1$. Then b = 0 and a is either 1 or $-(2^n - 1)$. For all k, $a \neq b$, which contradicts the assumption that LC(C) and RC(C) are not link-disjoint. ν

Lemma 4: If C_1 and C_2 are LDHC's in Q_n, then $LC(C_1)$, $RC(C_1)$, $LC(C_2)$ and $RC(C_2)$ are LDHC's in Q_{2n}.

Proof: By Lemma 3, we already know that $LC(C_1)$ and $RC(C_1)$ are link-disjoint, and that $LC(C_2)$ and $RC(C_2)$ are link-disjoint. Suppose u is a node in Q_{2n}. If the links incident to u in $LC(C_1)$ toggle two of the first n bits, and the links incident to u in $LC(C_2)$ toggle two of the last n bits, there will be no link conflict. If the links incident to u in $LC(C_1)$ and $LC(C_2)$ both toggle two of the first n bits, the links incident to u in $LC(C_1)$ will set the first n bits to the nodes adjacent to u in C_1, and the links incident to u in $LC(C_2)$ will set the first n bits to the nodes adjacent to u in C_2. Since C_1 and C_2 are link-disjoint, there will be no link conflict. A similar argument applies to the case where the links incident to u in $LC(C_1)$ and $LC(C_2)$ both toggle two of the last n bits. A similar argument shows that $LC(C_1)$ and $RC(C_2)$ are link-disjoint. ν

Theorem 1: If there are n / 2 LDHC's in Q_n, then there are m / 2 LDHC's in Q_m, where $m = n * 2^k$, $k > 1$.

Proof: Let $C_1, C_2, \ldots, C_{n/2}$ be the LDHC's in Q_n. $LC(C_1)$, $RC(C_1)$, $LC(C_2)$, $RC(C_2)$, ..., $LC(C_{n/2})$, $RC(C_{n/2})$ are the n LDHC's in Q_{2n}. These cycles are link-disjoint by Lemma 4, and Hamilton cycles by Lemmas 1 and 2. Repeat the same process to generate m / 2 LDHC's in Q_m. ν

Since Figure 1 showed n / 2 LDHC's for Q_4, there are n / 2 LDHC's for Q_n, when $n = 2^k$, $k > 1$, by Theorem 1. We have a construction for 3 LDHC's for Q_6, which we omit due to lack of space. Therefore there are n / 2 LDHC's for Q_n, when $n = 3 * 2^k$, $k > 1$.

III. Generating Link-Disjoint Spanning Subnetworks (LDSS's) of Q_n

Suppose C_1, C_2 and C_3 are 3 LDHC's of Q_n, and that e_1, e_2, ..., e_n are the links of C_3. If C_3 was disassembled, its odd-numbered links given to C_1 and its even-numbered links given to C_2, C_1 and C_2 would become LDSS's of degree 3, possibly with greatly reduced diameter. Figure 4 illustrates this procedure, and Table 2 gives the results of an experiment conducted on three LDHC's of Q_6. By repeating this procedure, four LDSS's of degree 3 could be constructed for Q_{12}, etc.

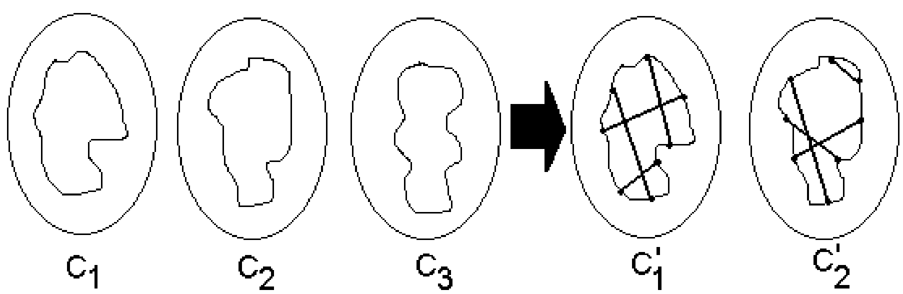

Figure 4. Changing 3 link-disjoint Hamilton cycles into 2 link-disjoint spanning subnetworks of degree 3

We will now generate two LDSS's of Q_n, where $n = 2^k$, $k > 2$. These spanning subnetworks will have degree (n / 2) - 2 and diameter n. Consider the two LDHC's for Q_4 shown in Figure 1. Figure 5 illustrates another representation of these two cycles. Let the cycle with the light links be called C_1 and the cycle with the dark links C_2. We now construct two degree 3 spanning subnetworks of

Disassembled Cycle	Odd # Links Given To...	Resulting Diameter	Even # Links Given To...	Resulting Diameter
C_3	C_1	11	C_2	9
C_3	C_2	10	C_1	8
C_2	C_3	10	C_1	14
C_2	C_1	11	C_3	9
C_1	C_2	12	C_3	9
C_1	C_3	12	C_2	14

Table 2. Improving the diameter by donating links of disassembled cycles

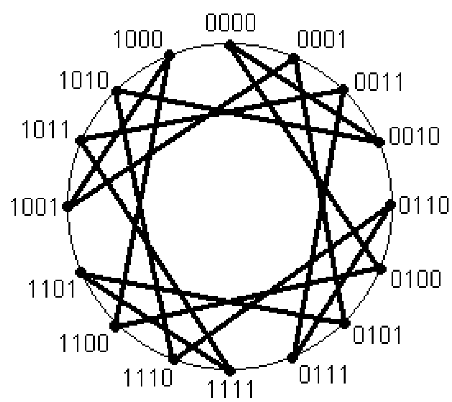

Figure 5. Another representation of two LDHC's of Q_4.

Q_4. The first subnetwork S_1 is the union of C_1 and the odd-numbered links of C_2. The second subnetwork S_2 is the union of C_1 and the even-numbered links of C_2. It can be verified that the diameter of both these subnetworks is 4. Figure 6 shows these two subnetworks. $LC(C_1)$, $RC(C_1)$, $LC(C_2)$ and $RC(C_2)$ are LDHC's of Q_8. We will disassemble $LC(C_2)$ and $RC(C_2)$ and give half the resulting links to $LC(C_1)$ and the other half to $RC(C_1)$ in the following manner. Let $w \mid x$ be the label of a node in $LC(C_1)$, where w and x are bit strings of length 4. If (w, y) is an odd-numbered link in C_2, then the link $(w \mid x, y \mid x)$ is a link of either $LC(C_2)$ or $RC(C_2)$. Let that link be removed and added to $LC(C_1)$. If (x, z) is an odd-numbered link of C_2, then the link $(w \mid x, w \mid z)$ is also a link of either $LC(C_2)$ or $RC(C_2)$. Let that link be removed and added to $LC(C_1)$. If (w, y) and (x, z) are

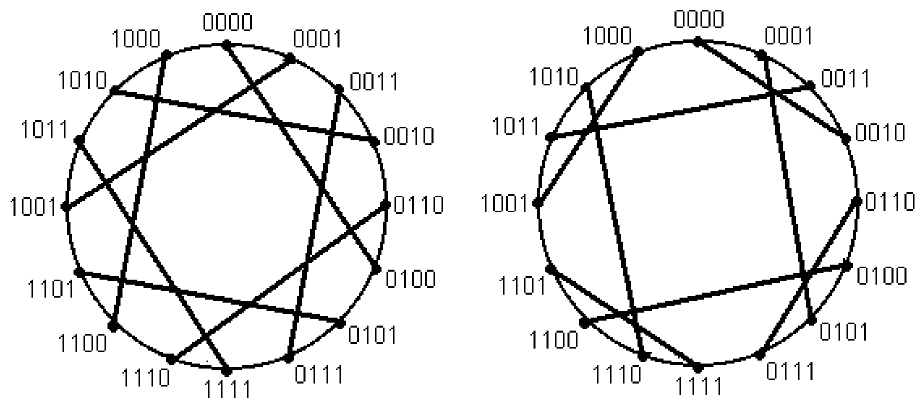

Figure 6. Two spanning subnetworks of Q_4 of degree 3 and diameter 4.

even-numbered links of C_2, then let the links (w | x, y | x) and (w | x, w | z) be removed from either $LC(C_2)$ or $RC(C_2)$ and added to $RC(C_1)$. Let $LC(C_1)'$ and $RC(C_1)'$ represent $LC(C_1)$ and $RC(C_1)$ with these additional links. $LC(C_1)'$ and $RC(C_1)'$ are now LDSS's of Q_8 of degree 4. Suppose we wish to route from w | x to y | z in $LC(C_1)'$. We can set w to x, and x to z by traversing S_1 in no more than 4 nodes each, for a total of 8 nodes. The diameter of $LC(C_1)'$ is 8. By a similar argument, the diameter of $RC(C_1)'$ is 8.

We will now show the general case. If C_0 is a Hamilton cycle in Q_n then let the *children* of C_0 be defined as $LC(C_0)$ and $RC(C_0)$. A Hamilton cycle C_1 in Q_{2n} is a *child* of C_0 if $C_1 = LC(C_0)$ or $C_1 = RC(C_0)$. A Hamilton cycle C_k in Q_{mn}, where $m = 2^k$, is a *descendent* of C *at level k* if there exists a sequence C_0, C_1, C_2, ..., C_k, where C_j is a child of C_{j-1}, for $1 \leq j \leq k$. Suppose C_k is a descendant of C_0 at level k, and that C_0, C_1, C_2, ..., C_k is the sequence demonstrating this. Let there be a bit string $w = w_1 w_2 \ldots w_k$ where $w_i = 0$ if $C_i = LC(C_{i-1})$ and $w_i = 1$ if $C_i = RC(C_{i-1})$. Let m be the integer equivalent of w for each descendant of C_0 at level k, and let $D(C_0, k, m)$ be the descendant of C_0 at level k with bit string w, $0 \leq m \leq 2^k - 1$.

Theorem 2: For $n = 2^k$, $k > 2$, Q_n contains two LDSS's with degree $(n / 4) + 2$, and diameter n.

Proof: Let C_a and C_b be descendants of C_1 at level k. Let $w_1 \mid w_2 \mid ... \mid w_{n/4}$ be the label of a node in C_a or C_b, where the w_i are bit strings of length 4 as before. For $1 \leq i \leq n / 4$, if (w_i, x) is an odd-numbered link of C_2, then the link $(w_1 \mid w_2 \mid ... \mid w_{n/4}, w_1 \mid w_2 \mid ... \mid w_{i-1} \mid x \mid w_{i+1} \mid ... \mid w_{n/4})$ is a link of some descendant of C_2 at level k. Let that link be removed and given to C_a. In the same way, for $1 \leq i \leq n / 4$, if (w_i, x) is an even-numbered link of C_2, then let the link $(w_1 \mid w_2 \mid ... \mid w_{n/4}, w_1 \mid w_2 \mid ... \mid w_{i-1} \mid x \mid w_{i+1} \mid ... \mid w_{n/4})$ be added to C_b. When routing from $w_1 \mid w_2 \mid ... \mid w_{n/4}$ to $x_1 \mid x_2 \mid ... \mid x_{n/4}$ in C_a, for $1 \leq i \leq n / 4$, w_i can be set to x_i by traversing S_1 in no more than 4 nodes. It takes no more than 4 nodes to set 4 bits in C_a, therefore the diameter of C_a is n. By a similar argument, the diameter of C_b is also n. ν

Theorem 3: If C is a Hamilton cycle in Q_n, then the union of LC(C) and RC(C) is a spanning subnetwork of Q_{2n} with degree 4 and diameter $2^{n.}$

Proof: When routing between two nodes, the first n bits can be arranged by no more than 2^{n-1} nodes by traversing RC(C), and the last n bits can be arranged by no more than 2^{n-1} nodes by traversing LC(C), for a diameter of 2^n. ν

Theorem 4: For $n = 2^k$, $k > 2$, Q_n contains n / 4 LDSS's of degree 4 and diameter $2^{n/2}$.

Proof: Let C_0 be the Hamilton cycle for Q_2, as in the proof for Theorem 2. The n / 4 LDSS's are the unions of D(C_0, k, m) and D(C_0, k, m + 1) for k ≥ 2 and even values of m. When this is the case, D(C_0, k, m) and D(C_0, k, m + 1) are the left and right children of some Hamilton cycle of $Q_{n/2}$, and the diameter of their union is $2^{n/2}$ by Theorem 3. ν

IV. Conclusions and Future Research

Given a hypercube of a particular size, we can construct 2, 4, ..., n / 2 LDSS's of a hypercube, which will allow multiple users to have access to a virtual copy of Q_n at one time. The more LDSS's are constructed, the larger the diameter. We conjecture there are many sets of LDHC's for Q_n, and it is a subject for future research to identify the set that yields the LDSS's with the best degree-diameter tradeoff. It is also a subject of research to identify an optimal strategy for combining LDHC's into LDSS's.

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